Chapter review

1 a $F = ma$

$$-\frac{k}{(x+2)^2} = 0.6a$$

The force is a function of x so use $a = v\frac{dv}{dx}$.

$$0.6v\frac{dv}{dx} = -\frac{k}{(x+2)^2}$$

$$0.6\int v\ dv = -\int \frac{k}{(x+2)^2} dx$$

Separate the variables and integrate.

$$0.3v^2 = \frac{k}{(x+2)} + c$$

$$x = 2, v = 8 \quad 0.3 \times 8^2 = \frac{k}{4} + c$$

$$x = 10, v = 2 \quad 0.3 \times 2^2 = \frac{k}{12} + c$$

Use the given information to obtain a pair of simultaneous equations in k and c.

$$\text{Subtract}: 0.3(8^2 - 2^2) = \frac{k}{4} - \frac{k}{12}$$

$$0.3 \times 60 = \frac{k}{6}$$

$$k = 0.3 \times 60 \times 6 = 108$$

Solve to find k.

b From above $0.3 \times 4 = \frac{k}{12} + c$

$$c = 1.2 - \frac{108}{12} = -7.8$$

Find c to complete the expression for v^2.

$$\therefore \quad 0.3v^2 = \frac{108}{(x+2)} - 7.8$$

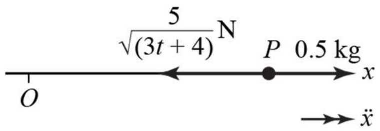
$$v = 0 \quad 0 = \frac{108}{x+2} - 7.8$$

$$7.8(x+2) = 108$$

$$x = \frac{108}{7.8} - 2 = 11.84\ldots$$

The distance OB is 11.8 m (3 s.f.).

2 a



$$F = ma$$

$$-\frac{5}{\sqrt{(3t+4)}} = 0.5\ddot{x}$$

$$\ddot{x} = -10(3t+4)^{-\frac{1}{2}}$$

$$\dot{x} = -\frac{10}{\frac{1}{2}\times 3}(3t+4)^{\frac{1}{2}} + c$$

$$t = 0 \quad \dot{x} = 12 \quad 12 = -\frac{20}{3}\sqrt{4} + c$$

$$c = 12 + \frac{40}{3} = \frac{76}{3}$$

$$\therefore\ \dot{x} = -\frac{20}{3}(3t+4)^{\frac{1}{2}} + \frac{76}{3}$$

b

$$x = -\frac{20}{3\times\frac{3}{2}\times 3}(3t+4)^{\frac{3}{2}} + \frac{76}{3}t + A$$

Integrate line above.

$$t = x = 0 \therefore A = \frac{40}{27}\times 4^{\frac{3}{2}} = \frac{320}{27}$$

$$P \text{ at rest} \Rightarrow \frac{76}{3} = \frac{20}{3}(3t+4)^{\frac{1}{2}}$$

Using result from **a**.

$$\left(\frac{76}{20}\right)^2 = 3t+4$$

$$t = \frac{1}{3}\left[\left(\frac{76}{20}\right)^2 - 4\right]$$

$$t = 3.48$$

When $t = 3.48$

$$x = -\frac{40}{27}\left(\frac{76}{20}\right)^3 + \frac{76}{3}\times 3.48 + \frac{320}{27}$$

$3t+4 = \left(\frac{76}{20}\right)^2$, so use the exact value here.

$$x = 18.72$$

P is 18.7 m from O (3 s.f.)

3 a $F = \dfrac{k}{x^2}$

when $x = R$, $F = \text{mg}$

$\therefore \dfrac{k}{R^2} = mg,\ k = mg\,R^2$

> When $x = R$, S is on the surface of the Earth and the force exerted by the Earth on S is mg.

b Force $= \dfrac{mg\,R^2}{x^2}$

$F = ma$

$-\dfrac{mgR^2}{x^2} = mv\dfrac{\mathrm{d}v}{\mathrm{d}x}$

$-\displaystyle\int \frac{gR^2}{x^2}\,\mathrm{d}x = \int v\,\mathrm{d}v$

$\dfrac{1}{2}v^2 = \dfrac{g\,R^2}{x} + c$

$x = 5R,\ v = 0\ c = \dfrac{-g\,R^2}{5R}$

$\therefore v^2 = 2g\dfrac{R^2}{x} - \dfrac{2g\,R^2}{5R}$

When $x = R$ $v^2 = 2g\dfrac{R^2}{R} - \dfrac{2g\,R^2}{5R}$

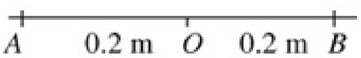
$v^2 = \dfrac{8Rg}{5}$

The speed of the spacecraft is

$\sqrt{\left(\dfrac{8Rg}{5}\right)} = 2\sqrt{\left(\dfrac{2Rg}{5}\right)}$

> The force is in the direction of *decreasing* x.

> The force is a function of x so use $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$.

4 

a amplitude $= 0.4 \div 2 = 0.2$ m

period $= 2 \times 2.5 = 5$s

$\therefore \quad \dfrac{2\pi}{\omega} = 5 \quad \omega = \dfrac{2\pi}{5}$

$v^2 = \omega^2(a^2 - x^2)$

$u^2 = \left(\dfrac{2\pi}{5}\right)^2 (0.2^2 - 0)$

$u = \dfrac{2\pi}{5} \times 0.2 = \dfrac{4\pi}{50}$(or 0.2513…)

$\therefore u = \dfrac{4\pi}{50}$(or 0.251 (3s.f.))

> Find a and ω from the given information.

> Now use $v^2 = \omega^2(a^2 - x^2)$ to find u.

4 b $x = a\sin\omega t$

$x = 0.2\sin\left(\frac{2\pi}{5}t\right)$

P is at the centre of oscillation when $t = 0$.

$\dot{x} = \frac{2\pi}{5}\times 0.2\cos\left(\frac{2\pi}{5}t\right)$

Differentiate x with respect to t to find $\dot{x}$.

$t = 3\quad \dot{x} = \frac{0.4\pi}{5}\cos\frac{6\pi}{5} = -0.2033$

When $t = 3$ P's speed is 0.203 m s^{-1} (3 s.f.).

Speed is positive.

c $x = a\sin\omega t$

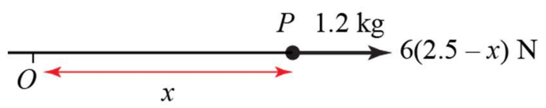
$x = 0.2\sin\left(\frac{2\pi}{5}t\right)$

$t = 3\quad x = 0.2\sin\left(\frac{6\pi}{5}\right) = -0.1175\ldots$

∴ Distance from A is 0.2 + 0.1175...
= 0.318 m (3 s.f.)

P is moving towards A when $t = 0$, so x is negative between O and B.

5



a $x = 2.5$

The acceleration (and therefore the resultant force) are zero when the speed is maximum.

b $F = ma$

$6(2.5 - x) = 1.2a$

$6(2.5 - x) = 1.2v\frac{dv}{dx}$

When the force is a function of x use $a = v\frac{dv}{dx}$.

$v\frac{dv}{dx} = 5(2.5 - x)$

$\int v\,dv = \int 5(2.5 - x)\,dx$

Separate the variables.

$\frac{1}{2}v^2 = 5\left(2.5x - \frac{x^2}{2}\right) + C$

Integrate. Don't forget the constant!

$x = 2.5\quad v = 8$

$\frac{1}{2}\times 8^2 = 5\left(2.5\times 2.5 - \frac{2.5^2}{2}\right) + C$

***a** gives the initial conditions.*

$C = 32 - 5\times\frac{2.5^2}{2} = 16.375$

$v^2 = 10\left(2.5x - \frac{x^2}{2}\right) + 2\times 16.375$

$v^2 = 25x - 5x^2 + 32.75$

6 a $x = 3\sin\left(\frac{\pi}{4}t\right)$

$\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

$\ddot{x} = -3\left(\frac{\pi}{4}\right)^2 \sin\left(\frac{\pi}{4}t\right)$

$\ddot{x} = -\left(\frac{\pi}{4}\right)^2 x$

$\therefore$ S.H.M.

Differentiate $x = 3\sin\left(\frac{\pi}{4}t\right)$ twice.

Obtain the equation of the form $\ddot{x} = -\omega^2 x$.

b amplitude $= 3m$

period $= \frac{2\pi}{\omega} = 2\pi \times \frac{4}{\pi} = 8\text{s}$

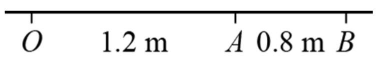
c From **a** $\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

$\Rightarrow$ maximum speed $= \frac{3\pi}{4}\text{ m s}^{-1}$

(or 2.36 m s^{-1} (3 s.f.))

Or use $v_{max} = a\omega$.

d



$x = 3\sin\left(\frac{\pi}{4}t\right)$

At A, $x = 1.2 \quad 1.2 = 3\sin\left(\frac{\pi}{4}t_a\right)$

$t_a = \frac{4}{\pi}\sin^{-1}\left(\frac{1.2}{3}\right)$

At B, $x = 2 \; t_b = \frac{4}{\pi}\sin^{-1}\left(\frac{2}{3}\right)$

Time $A \to B = \frac{4}{\pi}\left[\sin^{-1}\left(\frac{2}{3}\right) - \sin^{-1}\left(\frac{1.2}{3}\right)\right]$

$= 0.4051$

The time to go directly from A to B is 0.405 s (3 s.f.)

Leave calculator work as late as possible.

7 **a** $\ddot{x} = -\omega^2 x$

When $x = 0.09$, $\ddot{x} = -1.5$

$-1.5 = -\omega^2 \times 0.09$

$\omega^2 = \frac{50}{3}$

$\omega = \sqrt{\frac{50}{3}} = 4.08$

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3}{50}} = 1.54$ s

b $v^2 = \omega^2\left(a^2 - x^2\right)$

$0.3^2 = \frac{50}{3}\left(a^2 - 0.09^2\right)$

Solve to give

$a = 0.116$

c To find the time from $O \to \frac{a}{2}$

$x = a\sin\omega t$

$\frac{a}{2} = a\sin 4.08t$

$\frac{1}{2} = \sin 4.08t$

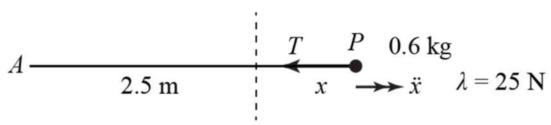
$4.08t = \frac{\pi}{6}$

$t = 0.128$ s

Time for one oscillation is 1.540 s

Therefore the time required is $1.540 - 4 \times 0.1283 = 1.03$ s

8



a $F = ma$

$-T = 0.6\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{25}{2.5}x = 10x$

$\therefore \quad 0.6\ddot{x} = -10x$

$\ddot{x} = -\frac{10}{0.6}x$

$\therefore$ S.H.M.

8 b $$\omega^2 = \frac{10}{0.6}$$

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.6}{10}} = 1.539\ldots$$

$$\text{period} = 1.54\,\text{s}\ (3\,\text{s.f.})$$

$$\text{amplitude} = (4 - 2.5)\,\text{m} = 1.5\,\text{m}$$

c $$x = a\cos\omega t$$

B is an end-point.

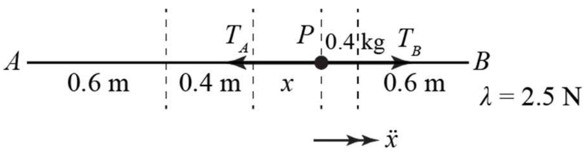
$$x = 1.5\cos\left(\sqrt{\frac{10}{0.6}}t\right)$$

$$x = -0.5\text{m} \quad -0.5 = 1.5\cos\left(\sqrt{\frac{10}{0.6}}t\right)$$

$$t = \sqrt{\frac{0.6}{10}}\cos^{-1}\left(-\frac{0.5}{1.5}\right) = 0.4680\ldots$$

P takes 0.468s to move 2 m from B (3 s.f.).

9



a $F = ma$

$$T_B - T_A = 0.4\ddot{x}$$

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T_A = \frac{2.5(0.4 + x)}{0.6}$$

$$T_B = \frac{2.5(0.4 - x)}{0.6}$$

$$\therefore \frac{2.5(0.4 - x)}{0.6} - \frac{2.5(0.4 + x)}{0.6} = 0.4\ddot{x}$$

$$-2 \times \frac{2.5x}{0.6} = 0.4\ddot{x}$$

$$\ddot{x} = -\frac{2 \times 2.5}{0.6 \times 0.4}x$$

Consider P to be attached to two strings, each of natural length 0.6 m and modulus 2.5 N.

$\therefore$ S.H.M.

b $$\omega^2 = \frac{2 \times 2.5}{0.6 \times 0.4} = \frac{5}{0.24}$$

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.24}{5}} = 1.376\ldots$$

The period is 1.38s (3 s.f.)

9 c $x = a\cos\omega t$

At D, $x = 1 - 0.85 = 0.15$

$$0.15 = 0.3\cos\left(\sqrt{\frac{5}{0.24}}t\right)$$

$$\sqrt{\frac{5}{0.24}}t = \cos^{-1}0.5$$

$$t = \sqrt{\frac{0.24}{5}}\cos^{-1}0.5$$

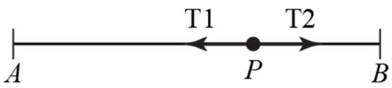
$$t = 0.2294\ldots$$

P takes 0.229 s (3 s.f.) to reach D.

For time from B (an end-point)

D and C are on the same side of the centre, so x is positive.

10 a



Hooke's law to AP:

$$T_1 = \frac{\lambda_1 x_1}{l_1} = \frac{20x_1}{5} = 4x_1$$

Hooke's law to BP:

$$T_2 = \frac{\lambda_2 x_2}{l_2} = \frac{18x_2}{3} = 6x_2$$

Resolving horizontally:

$T_1 = T_2$

So $4x_1 = 6x_2$

$$x_1 = \frac{3x_2}{2}$$

Length $AB = 12$

$5 + 3 + x_1 + x_2 = 12$

$x_1 + x_2 = 4$

$$\frac{3x_2}{2} + x_2 = 4$$

$$\frac{5x_2}{2} = 4$$

So $x_2 = \frac{8}{5}$ and $x_1 = \frac{12}{5}$

Hence the AP spring extension is 2.4 m and the PB spring extension is 1.6 m.

10 b Consider P displaced a distance x to the right of its equilibrium position.
Equation of motion of P:
$$T_2 - T_1 = m\ddot{x}$$
$$\frac{18(1.6-x)}{3} - \frac{20(2.4+x)}{5} = 0.4\ddot{x}$$
$$9.6 - 6x - 9.6 - 4x = 0.4\ddot{x}$$
$$-10x = 0.4\ddot{x}$$
$$\ddot{x} = -25x$$
Hence P oscillates with SHM.

c $\ddot{x} = -25x$
$$\omega^2 = 25 \Rightarrow \omega = 5$$

$$x = a\sin\omega t$$
$$x = a\sin 5t$$

Time from $x = 0$ to $x = 0.4$ is given by

$$0.4 = a\sin 5t$$
$$\frac{0.4}{a} = \sin 5t$$
$$t = \frac{1}{5}\arcsin\left(\frac{0.4}{a}\right)$$

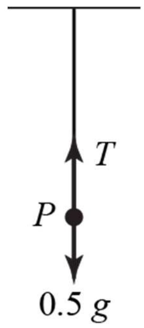
So P stays within 0.4 m of the equilibrium position for $\frac{4}{5}\arcsin\left(\frac{0.4}{a}\right)$ seconds.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$$
So $\frac{T}{3} = \frac{2\pi}{15}$

$$\therefore \frac{4}{5}\arcsin\left(\frac{0.4}{a}\right) = \frac{2\pi}{15}$$
$$\arcsin\left(\frac{0.4}{a}\right) = \frac{10\pi}{60} = \frac{\pi}{6}$$
$$\frac{0.4}{a} = 0.5$$
$$a = \frac{4}{5} = 0.8$$

Initial speed of P
$$= v_{max} = \omega a \text{ I}$$
$$= 5 \times \frac{4}{5}$$
$$= 4 \text{ m s}^{-1}$$

11 a



In equilibrium:

$R(\uparrow)\ T = 0.5g$

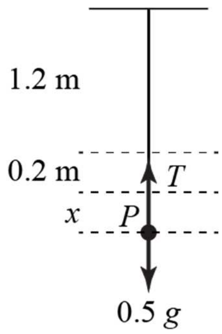
Hooke's Law : $T = \dfrac{\lambda x}{l}$

$$0.5g = \frac{\lambda \times 0.2}{1.2}$$

$$\lambda = 0.5 \times \frac{1.2}{0.2}$$

$$\therefore \lambda = 3g \quad (\text{or } 29.4)$$

b



For oscillations:

$$F = ma$$

$$0.5g - T = 0.5\ddot{x}$$

Hooke's Law : $T = \dfrac{\lambda x}{l}$

$$T = \frac{3g(0.2 + x)}{1.2}$$

$$\therefore 0.5g - \frac{3g(0.2+x)}{1.2} = 0.5\ddot{x}$$

$$\ddot{x} = -\frac{3g}{0.5 \times 1.2}x = -5gx$$

$\therefore$ S.H.M.

Of form $\ddot{x} = \omega^2 x$.

c

$$\omega^2 = 5g$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5g}} = 0.8975...$$

The period is 0.898 s (3 s.f.).

From $\ddot{x} = -5gx$.

d String becomes slack when $x = -0.2$ m.

amplitude $= 0.35$ m

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = 5g(0.35^2 - 0.2^2)$$

$$v = 2.010...$$

The speed is 2.01 m s^{-1} (3 s.f.).

Use the exact value for ω^2.

e $v^2 = u^2 + 2as$

$$0 = 2.010^2 - 2 \times 9.8s$$

$$s = \frac{2.010^2}{2 \times 9.8} = 0.2061...$$

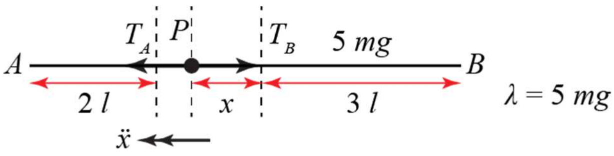
Distance above $O = 0.2 + 0.2061...$

$= 0.406$ m (3 s.f.)

Once the string is slack the particle moves freely under gravity.

The particle is 0.2 m above O when the string becomes slack.

12



a Hooke's Law : $T = \frac{\lambda x}{l}$

$$T_A = \frac{5mg(l-x)}{2l}$$

$$T_B = \frac{5mg(l+x)}{2l}$$

Consider the particle to be attached to two strings, *AP* and *PB*, both with natural length $2l$ and modulus 5 *mg*.

$$F = ma$$

$$T_A - T_B = m\ddot{x}$$

$$\frac{5mg(l-x)}{2l} - \frac{5mg(l+x)}{2l} = m\ddot{x}$$

$$-\frac{5mgx}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{5gx}{l}$$

$\therefore$ S.H.M.

b $\omega^2 = \frac{5g}{l}$ period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{5g}}$

The period is $2\pi\sqrt{\left(\frac{l}{5g}\right)}$

c amplitude $= \frac{3l}{4}$

Find the amplitude from the given information.

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{\text{max}} = \omega a = \sqrt{\frac{5g}{l}} \times \frac{3l}{4} = \frac{3}{4}\sqrt{5gl}$$

Maximum speed when $x = 0$.

The maximum speed is $\frac{3}{4}\sqrt{5gl}$.

Challenge

a The equation of motion for mass m is given by

$$ma = -\frac{mMg}{(R+x)^2}$$

$$a = -\frac{Mg}{(R+x)^2}$$

$$v\frac{dv}{dx} = -\frac{Mg}{(R+x)^2}$$

The mass reaches its maximum height H when $v = 0$.
Separating the variables and integrating:

$$\int_u^0 v dv = -\int_0^H \frac{MG}{(R+x)^2}dx$$

$$\left[\frac{v^2}{2}\right]_u^0 = \left[\frac{MG}{(R+x)}\right]_0^H$$

$$-\frac{u^2}{2} = \frac{MG}{R+H} - \frac{MG}{R}$$

$$-\frac{u^2}{2} = MG\left(\frac{1}{R+H} - \frac{1}{R}\right)$$

$$-\frac{u^2}{2} = MG\left(\frac{R-(R+H)}{R(R+H)}\right)$$

$$-\frac{u^2}{2} = -MG\left(\frac{H}{R(R+H)}\right)$$

$$\frac{u^2}{2} = MG\left(\frac{H}{R(R+H)}\right)$$

$$2MGH = u^2(R^2 + RH)$$

$$2MGH - RHu^2 = u^2R^2$$

$$H(2MG - Ru^2) = u^2R^2$$

$$H = \frac{u^2R^2}{2MG - Ru^2}$$

$$H = \frac{Ru^2}{\frac{2MG}{R} - u^2}$$

b As $u^2 \to \frac{2MG}{R}$, $H \to \infty$

The escape velocity is therefore given by

$$u = \sqrt{\frac{2MG}{R}}$$

$$u = \sqrt{\frac{2 \times 5.98 \times 10^{24} \times 6.7 \times 10^{-11}}{6.4 \times 10^6}}$$

$$u = 1.12 \times 10^4 \text{ m s}^{-1}$$